On the classification of binary self-dual $[44, 22, 8]$ codes with an automorphism of order 3 or 7

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Abstract

All binary self-dual $[44, 22, 8]$ codes with an automorphism of order 3 or 7 are classified. In this way we complete the classification of extremal self-dual codes of length 44 having an automorphism of odd prime order.

1 Introduction

Let $\mathbb{F}_q^n$ be the n-dimensional vector space over the field $\mathbb{F}_q$ of q elements. A *linear* $[n, k]$ *code* C is a k-dimensional subspace of $\mathbb{F}_q^n$. The elements of C are called *codewords*. The *weight* of a vector $v \in \mathbb{F}_q^n$ (denoted by $\mathrm{wt}(v)$) is the number of its non-zero coordinates. The *minimum weight* d of C is the smallest weight among all nonzero weights of codewords of C; a code C with minimum weight d is called an $[n, k, d]$ code. A matrix whose rows form a basis of C is called a *generator matrix* of this code. The weight enumerator $W(y)$ of a code C is given by $W(y) = \sum_{i=0}^{n} A_i y^i$ where A_i is the number of codewords of weight i in C. Two binary codes are called *equivalent* if one can be obtained from the other by a permutation of coordinates. The permutation $\sigma \in S_n$ is an *automorphism* of C, if $C = \sigma(C)$ and the set of all automorphisms of C forms a group called the *automorphism group* of C, which is denoted by $Aut(C)$ in this paper.

Let $(u, v) \in \mathbb{F}_q$ for $u, v \in \mathbb{F}_q^n$ be an inner product in $\mathbb{F}_q^n$. The *dual code* of an $[n, k]$ code C is $C^{\perp} = \{u \in \mathbb{F}_q^n \mid (u, v) = 0 \text{ for all } v \in C\}$ and $C^{\perp}$ is a linear $[n, n-k]$ code. If $C \subseteq C^{\perp}$, C is termed *self-orthogonal*, and if $C = C^{\perp}$, C is self-dual. We call a binary code *self-complementary* if it contains the all-ones vector. Every binary self-dual code is self-complementary. If $u = (u_1, \cdots, u_n)$, $v = (v_1, \cdots, v_n) \in \mathbb{F}_2^n$ then $(u, v) = \sum_{i=1}^{n} u_i v_i \in \mathbb{F}_2$. It was shown in [15] that the minimum weight d of a binary self-dual code of length n is bounded by $d \leq 4[n/24] + 4$, unless $n \equiv 22 \pmod{24}$ when $d \leq 4[n/24] + 6$. We call a self-dual code meeting this upper bound *extremal*.

In this paper, we consider extremal binary self-dual $[44, 22, 8]$ codes. All the odd primes p dividing the order of the automorphism group of such a code are 11, 7, 5, and 3 [20]. The codes with automorphisms of order 11 and 5 are classified in [20], [21], [3], and [4]. Unfortunately we noticed that there are some omissions in the classification of the codes with automorphisms of order 7 given in [16]. That's why we focus on the automorphisms of orders 3 and 7, and we complete the classification of $[44, 22, 8]$ self-dual codes having an automorphism of odd prime order.

As in the case of binary self-dual $[42, 21, 8]$ codes with an automorphism of order 3, there are five different possibilities for the number of independent cycles in the decomposition of the automorphism, namely 6, 8, 10, 12, and 14 [5]. Codes with automorphisms of order 3 with 6 and 14 independent 3-cycles are considered but not classified in [4] and [17], respectively. In this paper, we give the classification of all self-dual $[44, 22, 8]$ codes having an automorphism of order 3 or 7. To do that we apply the method for constructing binary self-dual codes via an automorphism of odd prime order developed in [8] and [18]. We give a short description of this method in Section 2. In Section 3 and Section 4 we classify the extremal self-dual codes of length 44 with an automorphism of order 3 and 7, respectively. In Section 5 we present the full classification of the self-dual $[44, 22, 8]$ codes having automorphisms of odd prime order, and offer some open problems.

The weight enumerators of the extremal self-dual codes of length 44 are known (see [7]):

$$W_{44,1}(y) = 1 + (44 + 4\beta)y^8 + (976 - 8\beta)y^{10} + (12289 - 20\beta)y^{12} + \dots$$

for $10 \leq \beta \leq 122$ and

$$W_{44,2}(y) = 1 + (44 + 4\beta)y^8 + (1232 - 8\beta)y^{10} + (10241 - 20\beta)y^{12} + \dots$$

for $0 \leq \beta \leq 154$.

Codes exist for $W_{44,1}$ when $\beta = 10, \dots, 68$, 70, 72, 74, 82, 86, 90, 122 and for $W_{44,2}$ when $\beta = 0, \dots, 56$, 58, …, 62, 64, 66, 68, 70, 72, 74, 76, 82, 86, 90, 104, 154 (see [9]).

2 Construction Method

Let C be a binary self-dual code of length $n = 44$ with an automorphism σ of prime order $p \geq 3$ with exactly c independent p-cycles and $f = 44 - cp$ fixed points in its decomposition. We may assume that

$$\sigma = (1, 2, \cdots, p)(p + 1, p + 2, \cdots, 2p) \cdots (p(c - 1) + 1, p(c - 1) + 2, \cdots, pc),$$

and say that σ is of type p-(c, f).

Denote the cycles of σ by $\Omega_1, \dots, \Omega_c$, and the fixed points by $\Omega_{c+1}, \dots, \Omega_{c+f}$. Let $F_\sigma(C) = \{v \in C \mid v\sigma = v\}$ and

$$E_\sigma(C) = \{v \in C \mid \mathrm{wt}(v|\Omega_i) \equiv 0 \pmod 2, i = 1, \cdots, c + f\},$$

where $v|\Omega_i$ is the restriction of v on Ω_i.

Theorem 1 [8] *The self-dual code C is a direct sum of the subcodes $F_\sigma(C)$ and $E_\sigma(C)$. These subcodes have dimensions $\frac{c+f}{2}$ and $\frac{c(p-1)}{2}$, respectively.*

Thus each choice of the codes $F_\sigma(C)$ and $E_\sigma(C)$ determines a self-dual code C. So for a given length all self-dual codes with an automorphism σ can be obtained.

We have that $v \in F_\sigma(C)$ if and only if $v \in C$ and v is constant on each cycle. Let $\pi : F_\sigma(C) \to \mathbb{F}_2^{c+f}$ be the projection map where if $v \in F_\sigma(C)$, $(\pi(v))_i = v_j$ for some $j \in \Omega_i$, $i = 1, 2, \ldots, c+f$.

Denote by $E_\sigma(C)^*$ the code $E_\sigma(C)$ with the last f coordinates deleted. So $E_\sigma(C)^*$ is a self-orthogonal binary code of length pc and dimension $c(p-1)/2$. For $v \in E_\sigma(C)^*$ we let $v|\Omega_i = (v_0, v_1, \cdots, v_{p-1})$ correspond to the polynomial $v_0 + v_1 x + \cdots + v_{p-1}x^{p-1}$ from $\mathcal{P}$, where $\mathcal{P}$ is the set of even-weight polynomials in $\mathcal{R}_p = \mathbb{F}_2[x]/\langle x^p - 1\rangle$. Thus we obtain the map $\varphi : E_\sigma(C)^* \to \mathcal{P}^c$. $\mathcal{P}$ is a cyclic code of length p with generator polynomial $x - 1$. It is known that $\varphi(E_\sigma(C)^*)$ is a submodule of the $\mathcal{P}$-module $\mathcal{P}^c$ [8, 19].

Theorem 2 [19] *A binary $[n, n/2]$ code C with an automorphism σ is self-dual if and only if the following two conditions hold:*

(i) $C_\pi = \pi(F_\sigma(C))$ is a binary self-dual code of length $c + f$,

(ii) for every two vectors u, v from $C_\varphi = \varphi(E_\sigma(C)^*)$ we have

$$u_1(x)v_1(x^{-1}) + u_2(x)v_2(x^{-1}) + \cdots + u_c(x)v_c(x^{-1}) = 0.$$

Let $x^p - 1 = (x-1)h_1(x)\ldots h_s(x)$, where $h_1, \ldots, h_s$ are irreducible binary polynomials. If $g_j(x) = (x^p-1)/h_j(x)$, and $I_j = \langle g_j(x)\rangle$ is the ideal in $\mathcal{R}_p$, generated by $g_j(x)$, then I_j is a fields with $2^{deg(h_j(x))}$ elements, $j = 1, 2, \ldots, s$, and $\mathcal{P} = I_1 \oplus I_2 \oplus \cdots \oplus I_s$ [13].

Lemma 3 [19] *Let $M_j = \{u \in \varphi(E_\sigma(C)^*) | u_i \in I_j, i = 1, 2, \ldots, c\}$, $j = 1, 2, \ldots, s$. Then*
1) M_j is a linear space over I_j, $j = 1, 2, \ldots, s$;
2) $C_\varphi = \varphi(E_\sigma(C)^*) = M_1 \oplus M_2 \oplus \cdots \oplus M_s$ (direct sum of $\mathcal{P}$-submodules);
3) If C is a self-dual code, then $\sum_{j=1}^{s} dim_{I_j} M_j = cs/2$.

In the case, when 2 is a primitive root modulo p, $\mathcal{P}$ is a field with 2^{p-1} elements and the following theorem holds

Theorem 4 [8] *Let 2 be a primitive root modulo p. Then the binary code C with an automorphism σ is self-dual iff the following two conditions hold:*

(i) C_π is a self-dual binary code of length $c + f$;

(ii) C_φ is a self-dual code of length c over the field $\mathcal{P}$ under the inner product $(u, v) = \sum_{i=1}^{c} u_i v_i^{(p-1)/2}$.

Let $\mathcal{B}$, respectively $\mathcal{D}$, be the largest subcode of C_π whose support is contained entirely in the left c, respectively, right f, coordinates. Suppose $\mathcal{B}$ and $\mathcal{D}$ have dimensions k_1 and k_2, respectively. Let $k_3 = k - k_1 - k_2$. Then there exists a generator matrix for C_π in the form

$$G_\pi = \begin{pmatrix} B & O \\ O & D \\ E & F \end{pmatrix} \tag{1}$$

where B is a $k_1 \times c$ matrix with $gen(\mathcal{B}) = [B\ O]$, D is a $k_2 \times f$ matrix with $gen(\mathcal{D}) = [O\ D]$, O is the appropriate size zero matrix, and $[E\ F]$ is a $k_3 \times n$ matrix. Let $\mathcal{B}^*$ be the code of length c generated by B, $\mathcal{B}_E$ the code of length c generated by the rows of B and E, $\mathcal{D}^*$ the code of length f generated by D, and $\mathcal{D}_F$ the code of length f generated by the rows of D and F. The following theorem is a modification of Theorem 2 from [12].

Theorem 5 *With the notations of the previous paragraph*

(i) $k_3 = rank(E) = rank(F)$,

(ii) $k_2 = k + k_1 - c = k_1 + \frac{f-c}{2}$, *and*

(iii) $\mathcal{B}_E^\perp = \mathcal{B}^*$ *and* $\mathcal{D}_F^\perp = \mathcal{D}^*$.

3 Extremal Self-Dual Codes of Length 44 with an Automorphism of Order 3

Using Theorem 4, as 2 is a primitive root modulo 3, $\mathcal{P}$ is a field with 4 elements. We have that $\mathcal{P} = \{0, e = x + x^2, w = 1 + x^2, w^2 = 1 + x\} \cong \mathbb{F}_4$ where e is the identity of $\mathcal{P}$. In this case C_φ is a (Hermitian) self-dual code of length c over the quaternary field $\mathcal{P}$ under the inner product $(u, v) = \sum_{i=1}^{c} u_i v_i^2$. Since the minimum distance of $E_\sigma(C)$ is at least 8, this Hermitian code should have minimum distance at least 4.

To classify the codes, we need additional conditions for equivalence. That's why we use the following theorem:

Theorem 6 [18] *The following transformations preserve the decomposition and send the code C to an equivalent one:*

(i) a permutation of the fixed coordinates;

(ii) a permutation of the 3-cycles coordinates;

(iii) a substitution $x \to x^2$ *in* C_φ *and*

(iv) a cyclic shift to each 3-cycle independently.

3.1 Codes with an automorphism of type 3-$(6, 26)$

The extremal self-dual $[44, 22, 8]$ codes having an automorphism of type 3-$(6, 26)$ are considered in [4] but the author didn't succeed to classify all codes. We do this classification now. Generator matrices of the codes C_φ and $E_\sigma(C)^*$ are presented in [4]. In the same paper, it is also proved that C_π is a binary self-dual $[32, 16, \geq 4]$ code with a generator matrix

$$G_\pi = \begin{pmatrix} 0 & D \\ I_6 & F \end{pmatrix}$$

where D generates a $[26, 10, 8]$ self-orthogonal code $\mathcal{D}^*$, and $\mathcal{D}_F$ is its dual code. The code $\mathcal{D}^*$ cannot be self-complementary (see [4]). According to [2], there are 1768 inequivalent $[26, 10, 8]$ self-orthogonal codes. Using as D generator matrices of those codes which are not self-complementary, we obtain the self-dual $[44, 22, 8]$ codes invariant under the given permutation. To test them for equivalence, we use the program Q-Extension [1]. The weight enumerators of the constructed codes are listed in Table 1.

Theorem 7 *There are exactly* 15621 *self-dual* $[44, 22, 8]$ *codes having an automorphism of type* 3-$(6, 26)$.

Table 1: Extremal self-dual $[44, 22, 8]$ codes having an automorphism of type 3-$(6, 26)$

β	14	15	16	17	18	19	20	21	22	23	24	25	26
W_1	-	-	-	-	4	16	33	31	59	62	82	79	72
W_2	11	26	58	201	342	433	505	462	677	685	717	599	611

β	27	28	29	30	31	32	33	34	35	36	37	38	39
W_1	47	72	48	51	51	68	54	64	39	54	38	38	29
W_2	463	490	452	485	654	724	674	851	558	530	430	438	327

β	40	41	42	43	44	45	46	47	48	49	50	51	52
W_1	32	32	28	35	66	49	51	41	40	33	39	29	33
W_2	328	238	194	120	140	72	89	43	85	13	46	5	27

β	53	54	55	56	57	58	59	60	61	62	63	64	65
W_1	17	24	8	18	4	15	4	7	1	5	1	2	3
W_2	5	21	6	11	-	15	1	6	1	7	-	2	-

β	66	67	68	70	72	74	76	82	86	90	104	122	154
W_1	5	2	1	2	1	2	-	1	1	1	-	1	-
W_2	1	-	1	1	3	4	2	2	1	1	1	-	1

3.2 Codes with an automorphism of type 3-$(8, 20)$

Up to equivalence, a unique Hermitian quaternary $[8, 4, 4]$ code exists (see [11]). So up to equivalence we have a unique subcode $E_\sigma(C)^*$. The code C_π is a binary self-dual $[28, 14, \geq 4]$ code with a generator matrix G_π given in (1) where B and D generate self-orthogonal

$[8, k_1, \geq 4]$ and $[20, k_1 + 6, \geq 8]$ codes, respectively. Since $0 \leq k_1 \leq 4$, $\mathcal{D}^*$ is a binary self-orthogonal $[20, 6 \leq k_2 \leq 10, \geq 8]$ code. All optimal binary self-orthogonal codes of length 20 are classified in [4]. There are exactly 23 inequivalent $[20, 6, 8]$ self-orthogonal codes, four inequivalent $[20, 7, 8]$ self-orthogonal codes, and a unique $[20, 8, 8]$ self-orthogonal code. Hence $k_1 \leq 2$.

In the case $k_1 = 2$ we obtain only two inequivalent extremal codes of length 44, both with weight enumerator $W_{44,2}$, respectively for $\beta = 68$ and $\beta = 76$. For $k_1 = 1$, there exist 52 self-dual $[44, 22, 8]$ codes, and for $k_1 = 0$, the inequivalent codes number 5399. Their weight enumerators are listed in Table 2.

Theorem 8 *There are exactly* 5453 *self-dual* $[44, 22, 8]$ *codes having an automorphism of type* 3-$(8, 20)$.

Remark: The extremal self-dual $[44, 22, 8]$ codes invariant under a permutation of type 3-(8,20) are considered independently in [10]. The author of that paper has classified all extremal self-dual codes which have an automorphism of order 3 with 8 independent 3-cycles.

Table 2: Extremal self-dual $[44, 22, 8]$ codes having an automorphism of type 3-$(8, 20)$

β	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
W_1	-	-	-	-	-	-	2	-	-	5	5	3	9	16	8
W_2	2	-	3	10	8	27	47	81	157	174	330	395	442	481	560
β	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
W_1	16	28	16	69	36	39	27	60	29	55	26	34	15	25	15
W_2	442	432	307	298	140	172	79	69	41	56	13	25	9	9	6
β	38	39	40	41	42	43	44	45	46	49	50	52	53	68	76
W_1	15	3	8	4	6	1	4	-	3	1	1	-	1	-	-
W_2	18	-	9	4	6	3	4	-	3	-	3	1	-	1	1

3.3 Codes with an automorphism of type 3-$(10, 14)$

In this case C_φ is a Hermitian self-dual $[10, 5, 4]$ code and by [11] is equivalent to either E_{10} or B_{10}. As in [5], we can fix the generator matrix of the subcode $E_\sigma(C)^*$ in the following two forms, respectively:

$$\begin{pmatrix} 011011011011000000000000000000 \\ 101101101101000000000000000000 \\ 000000011011011011000000000000 \\ 000000101101101101000000000000 \\ 000000000000011011011011000000 \\ 000000000000101101101101000000 \\ 000000000000000000011011011011 \\ 000000000000000000101101101101 \\ 011000011000011000011000101110 \\ 101000101000101000101000110011 \end{pmatrix} \text{ and } \begin{pmatrix} 011011011011000000000000000000 \\ 101101101101000000000000000000 \\ 000011101110011000000000000000 \\ 000101110011101000000000000000 \\ 000000000000000011011011011000 \\ 000000000000000101101101101000 \\ 000000000000000000011101110011 \\ 000000000000000000101110011101 \\ 000011110101000000011110101000 \\ 000101011110000000101011110000 \end{pmatrix}.$$

The code C_π has parameters $[24, 12, \geq 4]$. There are exactly thirty inequivalent such codes, namely E_8^3, $E_{16} \oplus E_8$, $F_{16} \oplus E_8$, E_{12}^2, and the indecomposable codes denoted by $A_{24}, B_{24}, \ldots, Z_{24}$ in [6]. All codes have minimum weight 4 except the extended Golay code G_{24} with minimum weight 8 and the code Z_{24} with minimum weight 6. We use the generator matrices of the codes given in [14]. For any weight 4 vector in C_π at most two nonzero coordinates may be fixed points. An examination of the vectors of weight 4 in the listed codes eliminates 23 of them. By investigation of all alternatives for a choice of the 3-cycle coordinates in the remaining codes G_{24}, R_{24}, U_{24}, W_{24}, X_{24}, Y_{24} and Z_{24} we obtain, up to equivalence, all possibilities for the generator matrix of the code C_π.

Let C_π be R_{24}. There is a unique possibility for the choice of the 3-cycle coordinates up to equivalence. The generator matrix of C_π in this case can be fixed in the form

$$G_\pi(R_{24}) = \begin{pmatrix} 1100000000\ 11000000000000 \\ 0110000000\ 01100000000000 \\ 0001100000\ 00011000000000 \\ 0000110000\ 00001100000000 \\ 0000001100\ 00000011000000 \\ 0000000110\ 00000001100000 \\ 0000000011\ 00000000110000 \\ 1001000000\ 10010000001111 \\ 1000001000\ 10000010001100 \\ 0000001110\ 00000000010110 \\ 1110000000\ 00000000000111 \\ 0001110000\ 00000000001110 \end{pmatrix}.$$

Let τ be a permutation of the ten cycle coordinates in $G_\pi(R_{24})$. Denote by C^τ the self-dual $[44, 22]$ code determined by C_φ and the matrix $\tau(G_\pi(R_{24}))$.

We consider the products of transformations (ii), (iii) and (iv) from Theorem 6 which preserve the quaternary code C_φ. Their permutation parts form a subgroup of the symmetric group S_{10} which we denote by L. Let $S = Stab(R_{24})$ be the stabilizer of the automorphism group of the code generated by $G_\pi(R_{24})$ on the set of the fixed points. It is easy to prove that if τ_1 and τ_2 are permutations from the group S_{10}, the codes C^{τ_1} and C^{τ_2} are equivalent iff the double cosets $S\tau_1 L$ and $S\tau_2 L$ coincide. In our case $Stab(R_{24}) = \langle (7, 8)(9, 10)\,, (7, 9, 10), (7, 9)(8, 10), (7, 10), (5, 6), (4, 6, 5), (2, 3), (1, 3, 2), (1, 4)(2, 5)(3, 6) \rangle$.

When $C_\varphi = B_{10}$ we found in [5] a subgroup of the group L generated by the permutations $(3, 4)(8, 9)$, $(1, 2)(3, 4)$, $(1, 3)(2, 4)$, $(6, 7)(8, 9)$, $(6, 8)(7, 9)$ and $(1, 6)(2, 7)(3, 8)(4, 9)(5, 10)$. So we obtain four $[44, 22, 8]$ self-dual codes: $C_{B_{10}}^{id}$, $C_{B_{10}}^{(567)}$, $C_{B_{10}}^{(36754)}$ and $C_{B_{10}}^{(368574)}$. These codes have weight enumerator $W_{44,1}$ with $\beta = 60, 33$, 30 and 21 and automorphism groups of orders $2^7 \cdot 3^4$, $2^4 \cdot 3^3$, 72 and 48, respectively.

When $C_\varphi = E_{10}$ the group $L = \langle (1, 3, 5, 7, 9)(2, 4, 6, 8, 10), (1, 2)(3, 4)\,, (1, 3)(2, 4), (9, 10) \rangle$. We obtain seven $[44, 22, 8]$ self-dual codes $C_{E_{10}}^\tau$ for $\tau \in \{$id, (4,5,6,7),(4,5,7)(6,9,8), $(2, 3, 5, 4)$, $(2, 3, 5, 4)(6, 7)$, $(2, 3, 5, 7, 4)(6, 9, 8)$, $(6, 7)\}$. These codes have also weight enumerator $W_{44,1}$ with $\beta = 42, 30$, 36, 24, 42, 30 and 21 and automorphism groups of orders $2^{10} \cdot 3$, 24, 192, 36, $2^7 \cdot 3^2$, again 24 and 720, respectively.

In this way from all the cases for C_π we constructed 1865 inequivalent $[44,22,8]$ self-dual codes with weight enumerator $W_{44,1}$ for $\beta = 10,\dots,52$, 54, 55, 60, 62, 65 and 6873 codes with weight enumerator $W_{44,2}$ for $\beta = 3,\dots,36$, 38, 42, 45, 46, 50 and 52. The calculations for these results were done with the GAP Version 4r4 software system and the program Q-Extension [1]. The results are summarized in Tables 3 and 4.

Table 3: Extremal self-dual $[44,22,8]$ codes having an automorphism of type 3-$(10,14)$

	$W_{44,1}$	$W_{44,2}$		$W_{44,1}$	$W_{44,2}$		$W_{44,1}$	$W_{44,2}$
G_{24}, B_{10}	3	12	U_{24}, E_{10}	74	49	Y_{24}, B_{10}	136	746
G_{24}, E_{10}	6	25	W_{24}, B_{10}	71	11	Y_{24}, E_{10}	456	2764
R_{24}, B_{10}	4	-	W_{24}, E_{10}	188	33	Z_{24}, B_{10}	71	541
R_{24}, E_{10}	7	-	X_{24}, B_{10}	161	224	Z_{24}, E_{10}	207	1824
U_{24}, B_{10}	29	19	X_{24}, E_{10}	459	635			

Theorem 9 *There are exactly* 8738 *inequivalent self-dual* $[44,22,8]$ *codes having an automorphism of type* 3-$(10,14)$.

Table 4: Extremal self-dual $[44,22,8]$ codes having an automorphism of type 3-$(10,14)$

β	3	4	5	6	7	8	9	10	11	12	13	14	15	16
W_1	-	-	-	-	-	-	-	1	2	11	49	63	25	114
W_2	1	3	31	31	93	143	183	377	428	560	622	552	755	510
β	17	18	19	20	21	22	23	24	25	26	27	28	29	30
W_1	97	51	159	134	71	157	99	63	129	81	49	90	61	41
W_2	411	585	270	223	321	145	96	176	35	71	64	32	13	57
β	31	32	33	34	35	36	37	38	39	40	41	42	43	44
W_1	55	31	28	41	21	16	22	21	11	14	11	10	12	4
W_2	7	23	16	11	3	8	-	9	-	-	-	4	-	-
β	45	46	47	48	49	50	51	52	54	55	60	62	65	
W_1	4	4	1	1	1	2	1	2	1	1	1	1	1	
W_2	1	1	-	-	-	1	-	1	-	-	-	-	-	

3.4 Codes with an automorphism of type 3-$(12,8)$

In this case C_φ is a quaternary Hermitian self-dual code of length 12 with minimum weight at least 4. There exist exactly five inequivalent quaternary self-dual $[12,6,4]$ codes, denoted by d_{12}, $2d_6$, $3d_4$, $e_6 \oplus e_6$, and $e_7 + e_5$ in [11].

The code C_π is a binary self-dual $[20,10,\geq 4]$ code. There are exactly seven such codes, namely $d_{12}+d_8$, $d_{12}+e_8$, d_{20}, d_4^5, $d_6^3+f_2$, $d_8^2+d_4$, and $e_7^2+d_6$ [6]. Each choice for the fixed

points can lead to a different subcode $F_\sigma(C)$. We have considered all possibilities for each of these seven codes, and found exactly 7 inequivalent codes for $d_{12}+d_8$, one code for $d_{12}+e_8$, one code for d_{20}, 10 codes for d_4^5, 26 codes for $d_6^3+f_2$, 18 codes for $d_8^2+d_4$, and 3 codes for $e_7^2+d_6$. Denote these codes by $H_{i,j}$, for $i=1,2\dots,7$.

By the method used in Section 3.3, considering the permutation parts of the products of transformations (ii), (iii) and (iv) from Theorem 6 and the stabilizer of the automorphism group of the codes $H_{i,j}$ on the fixed points, we classified all codes up to equivalence. There are exactly 122787 inequivalent codes. Their weight enumerators are of type $W_{44,1}$ for $\beta=$ 10, ..., 68, 70, 72, 74, 82, 86, 90, 122 and of type $W_{44,2}$ for $\beta=0$, ..., 56, 58, ..., 62, 64, 66, 68, 70, 72, 74, 76, 82, 86, 90, 104, 154. The values obtained for β are listed in Table 5.

Theorem 10 *There are exactly* 122787 *inequivalent self-dual* $[44,22,8]$ *codes having an automorphism of type* 3-$(12,8)$.

Table 5: Extremal self-dual $[44,22,8]$ codes having an automorphism of type 3-$(12,8)$

β	0	1	2	3	4	5	6	7	8	9	10	11
W_1	-	-	-	-	-	-	-	-	-	-	789	556
W_2	7	151	594	1434	2178	3468	5793	7034	6881	9434	10031	6906
β	12	13	14	15	16	17	18	19	20	21	22	23
W_1	313	1915	1072	655	2141	1105	912	1770	1029	736	1338	666
W_2	8502	7975	5072	4805	5111	2549	2552	2438	1692	1176	1609	778
β	24	25	26	27	28	29	30	31	32	33	34	35
W_1	642	731	511	382	568	286	286	263	236	161	179	99
W_2	773	745	532	311	484	204	242	169	217	65	176	32
β	36	37	38	39	40	41	42	43	44	45	46	47
W_1	126	87	88	55	69	38	52	28	48	17	32	10
W_2	73	42	68	30	44	29	30	21	21	9	26	10
β	48	49	50	51	52	53	54	55	56	57	58	59
W_1	18	7	19	9	15	5	7	3	11	4	9	5
W_2	14	7	17	3	15	4	9	6	13	-	11	1
β	60	61	62	63	64	65	66	67	68	70	72	74
W_1	3	1	2	1	2	3	4	2	1	2	1	2
W_2	6	1	5	-	2	-	1	-	1	1	3	4
β	76	82	86	90	104	122	154					
W_1	-	1	1	1	-	1	-					
W_2	2	2	1	1	1	-	1					

3.5 Codes with an automorphism of type 3-$(14, 2)$

The code C_π in this case is a self-dual $[16, 8, 4]$ code. There are exactly three such codes, namely d_8^2, d_{16}, and e_8^2 [6]. We consider their generator matrices in the form

$$G_1 = gen(d_8^2) = \begin{pmatrix} 1000000011100000 \\ 0100000011010000 \\ 0010000000001110 \\ 0001000000001101 \\ 0000100011001011 \\ 0000010011000111 \\ 0000001010111100 \\ 0000000101111100 \end{pmatrix}, \quad G_2 = gen(d_{16}) = \begin{pmatrix} 1111000000000000 \\ 0011110000000000 \\ 0000111100000000 \\ 0000001111000000 \\ 0000000011110000 \\ 0000000000111100 \\ 0000000000001111 \\ 0101010101010101 \end{pmatrix},$$

and $G_3 = gen(e_8^2) = \begin{pmatrix} HO \\ OH \end{pmatrix}$, where $H = (I_4|J + I_4)$, J is the all-ones 4×4 matrix and O is the 8×8 zero matrix. We have to consider permutations on these generator matrices that can lead to different subcodes $F_\sigma(C)$. From all possibilities for each of these codes we have found exactly 7 different cases for C_π which can produce inequivalent codes C, namely G_1, $G_1^{(1,16)}$, $G_1^{(3,16)}$, G_2, $G_2^{(1,16)}$, G_3, and $G_3^{(1,16)}$.

The code C_φ is a quaternary Hermitian self-dual $[14, 7, 4]$ code. There are exactly 10 such codes, namely d_{14}, $2e_7$, $d_8 + e_5 + f_1$, $2e_5 + d_4$, $d_8 + d_6$, $2d_6 + f_2$, $d_6 + 2d_4$, $3d_4 + f_2$, $2d_4 + 1_8$, and q_{14} [11].

Again, considering the permutation parts of the products of transformations (ii), (iii) and (iv) from Theorem 6, and the stabilizer of the automorphism group of the codes C_π on the fixed points, we classified all codes up to equivalence.

When $C_\pi = d_{16}$ all codes have weight enumerators $W_{44,1}$ for β=11, 14, 17, 20, 23, 26, 29, 32, 35, 38, 41, 44, 53, 62, and 65. When $C_\pi = e_8^2$ the weight enumerators are $W_{44,1}$ for β=10, 13, 16, 19, 22, 25, 28, 31, 34, 37, 40, 43, 46, 49, 52, and 58. Lastly, when $C_\pi = d_8^2$ we constructed codes with weight enumerator $W_{44,2}$ for β =1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 25, 26, 28, 29, 31, 32, 34, 35, 37, 38, 40, 41, 43, 44, 46, 52, and 55. The total number of all self-dual $[44, 22, 8]$ codes, having an automorphism of type 3-$(14, 2)$ is 243927. The results are presented in Tables 6 and 7.

Theorem 11 *There are exactly* 243927 *inequivalent self-dual* $[44, 22, 8]$ *codes having an automorphism of type* 3-$(14, 2)$.

3.6 All self-dual $[44, 22, 8]$ codes with an automorphism of order 3

Here we summarize all obtained results for the extremal self-dual codes of length 44 having an automorphism of order 3. To test the codes for equivalence, we used the program Q-Extension. The classification result is given in the following theorem.

Table 6: Extremal self-dual $[44, 22, 8]$ codes having an automorphism of type 3-$(14, 2)$

	d_{14}	$2e_7$	$d_8 + e_5 + f_1$	$2e_5 + d_4$	$d_8 + d_6$
d_{16}	7	33	66	26	144
e_8^2	9	20	77	26	197
d_8^{2+}	114	876	2907	490	6148
	$2d_6 + f_2$	$d_6 + 2d_4$	$3d_4 + f_2$	$2d_4 + 1_8$	q_{14}
d_{16}	573	384	2040	1663	1191
e_8^2	735	496	2830	2225	1561
d_8^{2+}	25841	14639	84081	60246	34520

Table 7: Extremal self-dual $[44, 22, 8]$ codes having an automorphism of type 3-$(14, 2)$

β	1	2	4	5	7	8	10	11	13	14
W_1	-	-	-	-	-	-	704	984	1912	1537
W_2	4565	4374	21709	15796	35653	26242	33236	22914	21064	14322
β	16	17	19	20	22	23	25	26	28	29
W_1	2006	1281	1447	1008	978	493	480	384	295	147
W_2	10879	6663	4407	3053	1866	992	621	521	344	152
β	31	32	34	35	37	38	40	41	43	44
W_1	123	124	98	29	17	54	21	8	9	18
W_2	109	88	85	19	16	24	14	9	4	2
β	46	49	52	53	55	58	62	65		
W_1	7	2	3	1	-	1	1	2		
W_2	4	-	4	-	2	-	-	-		

Theorem 12 *There are exactly* 394916 *inequivalent self-dual* $[44, 22, 8]$ *codes having an automorphism of order* 3*.*

We list the number of the codes with different weight enumerators in Table 8. For $\beta \geq 67$, all codes have simultaneously automorphisms of type 3-(12,8) and also automorphisms of type 3-(6,26). This proves that the orders of the automorphism groups of these codes are multiples of 9. We give these orders in Table 13. All codes with $\beta \geq 63$ have automorphisms of type 3-(6,26). All seven codes with $\beta = 0$ have automorphisms of type 3-(12,8). The full automorphism group for four of them is the cyclic group of order 3, and the other three codes have automorphism groups of order 12.

Table 8: All extremal self-dual $[44, 22, 8]$ codes having an automorphism of order 3

β	0	1	2	3	4	5	6	7	8	9	10
W_1	-	-	-	-	-	-	-	-	-	-	1487
W_2	7	4713	4968	1435	23881	19271	5824	42768	33242	9617	43614
β	11	12	13	14	15	16	17	18	19	20	21
W_1	1539	324	3860	2659	680	4248	2471	972	3385	2182	851
W_2	30231	9070	29668	19954	5666	16669	9965	3804	7898	5880	2440
β	22	23	24	25	26	27	28	29	30	31	32
W_1	2523	1327	807	1428	1080	512	1051	558	431	515	504
W_2	4798	2963	2095	2250	1985	978	1465	870	849	965	1048
β	33	34	35	36	37	38	39	40	41	42	43
W_1	266	396	201	213	176	205	95	131	86	92	79
W_2	761	1082	606	609	477	526	348	366	258	221	134
β	44	45	46	47	48	49	50	51	52	53	54
W_1	115	66	86	47	55	39	55	34	43	18	28
W_2	151	73	102	44	87	15	51	5	30	5	23
β	55	56	57	58	59	60	61	62	63	64	65
W_1	8	25	5	20	5	7	1	6	1	2	3
W_2	6	15	-	17	1	6	1	7	-	2	-
β	66	67	68	70	72	74	76	82	86	90	104
W_1	5	2	1	2	1	2	-	1	1	1	-
W_2	1	-	1	1	3	4	2	2	1	1	1
β	122	154									
W_1	1	-									
W_2	-	1									

4 Extremal Self-Dual Codes of Length 44 with an automorphism of order 7

If σ is an automorphism of a binary self-dual $[44, 22, 8]$ code of order 7, then σ is of type 7-$(3, 23)$ or 7-$(6, 2)$ [9].

Let $h_1(x) = (x^3 + x + 1)$ and $h_2(x) = (x^3 + x^2 + 1)$. As $x^7 - 1 = (x - 1)h_1(x)h_2(x)$, we have $\mathcal{P} = I_1 \oplus I_2$, where I_j is an irreducible cyclic code of length 7 with parity-check polynomial $h_j(x)$, $j = 1, 2$. According Lemma 3, $C_\varphi = M_1 \oplus M_2$, where $M_j = \{u \in C_\varphi \mid u_i \in I_j, i = 1, \dots, c\}$ is a linear code over the field I_j, $j = 1, 2$, and $\dim_{I_1} M_1 + \dim_{I_2} M_2 = c$. The polynomials $e_1 = x^4 + x^2 + x + 1$ and $e_2 = x^6 + x^5 + x^3 + 1$ generate the ideals I_1 and I_2 defined above. Any nonzero element of $I_j = \{0, e_j, xe_j \dots, x^6 e_j\}, j = 1, 2$ generates a binary cyclic $[7, 4, 3]$ code. Since the minimum weight of the code C is 8, every vector of C_φ must contain at least two nonzero coordinates. Hence the minimum weight of M_j is at least 2, $j = 1, 2$.

The transformation $x \to x^{-1}$ interchanges e_1 and e_2. The orthogonal condition (ii) from Theorem 2 implies that once chosen, M_1 determines M_2 and the whole C_φ. So we can

assume, without loss of generality, that $\dim_{I_1} M_1 \le \dim_{I_2} M_2$, and we can examine only M_1.

4.1 Codes with an automorphism of type 7-$(3,23)$

Let C be a binary self-dual $[44,22,8]$ code having an automorphism of type 7-$(3,23)$. Then we have $\dim_{I_1} M_1 + \dim_{I_2} M_2 = 3$. Since the minimum weight of M_2 is at least 2, we have $1 \le \dim_{I_1} M_1 \le \dim_{I_2} M_2 \le 2$. Hence $\dim_{I_1} M_1 = 1$ and $\dim_{I_2} M_2 = 2$. Then M_2 is an MDS $[3,2,2]$ code over the field I_2 and according to condition (ii) from Theorem 2, $M_1 = \langle (e_1,e_1,e_1)\rangle$ and $M_2 = \langle (e_2,e_2,0),(0,e_2,e_2)\rangle$.

In this case C_π is a binary self-dual code of length 26. If $v = (1100\ldots 0) \in C_\pi$ then $\pi^{-1}(v) + (\phi^{-1}(e_2,e_2,0),00\ldots 0)$ will be a codeword from C of weight 6 which contradicts the minimum weight of C. Hence in the notations of Theorem 5, $k_1 = 0, k_2 = 10, k_3 = 3$, and *gen* $C_\pi = \begin{pmatrix} 0 & D \\ E & F \end{pmatrix}$, where the matrix D generates a $[23,10,\ge 8]$ binary self-orthogonal code. There are three such codes and their generator matrices are given in [2]. We take $E = I_3$, and we determine the matrix F using the condition (iii) of Theorem 5. For each of the three codes there is a unique possibility for the matrix F, up to equivalence. We obtain the codes $C_{7,1}$ with weight enumerator $W_{44,1}$ for $\beta = 122$, $C_{7,2}$ with weight enumerator $W_{44,2}$ for $\beta = 104$, and $C_{7,3}$ with weight enumerator $W_{44,2}$ for $\beta = 154$. The orders of their automorphism groups are $3251404800 = 2^{15}\cdot 3^4 \cdot 5^2 \cdot 7^2$, $116121600 = 2^{13}\cdot 3^4\cdot 5^2\cdot 7$, and $786839961600 = 2^{16}\cdot 3^4\cdot 5^2\cdot 7^2\cdot 11^2$, respectively. All of these codes have automorphisms of order 5 and are known from [3].

Theorem 13 *There are exactly three inequivalent binary* $[44,22,8]$ *codes having an automorphism of type* 7-$(3,23)$.

4.2 Codes with an automorphism of type 7-$(6,2)$.

Let C be a binary self-dual $[44,22,8]$ code having an automorphism of type 7-$(6,2)$. Now C_π is a binary $[8,4]$ self-dual code equivalent either to C_2^4 or the extended Hamming code E_8, generated by the matrices $G_1 = (I_4|I_4)$ and $G_2 = (I_4|J+I_4)$, respectively where I_4 is the 4×4 identity matrix and J is the all-ones 4×4 matrix.

In this case $\dim_{I_1} M_1 + \dim_{I_2} M_2 = 6$ and $1 \le \dim_{I_1} M_1 \le \dim_{I_2} M_2 \le 5$. Hence $\dim_{I_1} M_1 = 1, 2$, or 3.

Case I: $\dim_{I_1} M_1 = 1, \dim_{I_2} M_2 = 5$. It follows that M_2 is an MDS $[6,5,2]$ code, and $M_1 = \langle (e_1,e_1,e_1,e_1,e_1,e_1)\rangle$. If $C_\pi = C_2^4$, then C_π contains a codeword $v = (v_1,00)$ such that $\mathrm{wt}(v_1) = 2$. Since M_2 is an MDS code, it contains a codeword w of weight 2 with the same support as v_1. But then the codeword $\pi^{-1}(v) + (\phi^{-1}(w),00) \in C$ has weight 6 - a contradiction. Therefore $C_\pi = E_8$. Fixing the codes M_1 and M_2 and considering all binary codes equivalent to E_8, we found only one $[44,22,8]$ code with weight enumerator $W_{44,1}$ for $\beta = 38$ and $|Aut(C)| = 8064$.

Case II: $\dim_{I_1} M_1 = 2, \dim_{I_2} M_2 = 4$. We can take

$$gen(M_1) = \left(\begin{array}{cccccc} e_1 & 0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ 0 & e_1 & \alpha_5 & \beta_1 & \beta_2 & \beta_3 \end{array} \right),$$

where $\alpha_i \in \{0, e_1\}, i = 1, \cdots, 5$, and $\beta_i \in I_1, i = 1, 2, 3$. Considering all such matrices we obtain nine possibilities such that the minimum weight of M_1 is ≥ 2, up to equivalence. Here $gen(M_1)$ is written in the form $(I_2|A)$, where A is one of the following matrices:

$$A_1 = \left(\begin{array}{cccc} e_1 & 0 & 0 & 0 \\ e_1 & e_1 & e_1 & e_1 \end{array} \right), \quad A_4 = \left(\begin{array}{cccc} e_1 & e_1 & 0 & 0 \\ e_1 & xe_1 & e_1 & e_1 \end{array} \right), \quad A_7 = \left(\begin{array}{cccc} e_1 & e_1 & e_1 & 0 \\ e_1 & xe_1 & x^2e_1 & e_1 \end{array} \right),$$

$$A_2 = \left(\begin{array}{cccc} e_1 & e_1 & 0 & 0 \\ 0 & 0 & e_1 & e_1 \end{array} \right), \quad A_5 = \left(\begin{array}{cccc} e_1 & e_1 & e_1 & 0 \\ 0 & e_1 & e_1 & e_1 \end{array} \right), \quad A_8 = \left(\begin{array}{cccc} e_1 & e_1 & e_1 & 0 \\ e_1 & xe_1 & x^3e_1 & e_1 \end{array} \right),$$

$$A_3 = \left(\begin{array}{cccc} e_1 & e_1 & 0 & 0 \\ 0 & e_1 & e_1 & e_1 \end{array} \right), \quad A_6 = \left(\begin{array}{cccc} e_1 & e_1 & e_1 & 0 \\ 0 & e_1 & xe_1 & e_1 \end{array} \right), \quad A_9 = \left(\begin{array}{cccc} e_1 & e_1 & e_1 & e_1 \\ e_1 & xe_1 & x^2e_1 & x^3e_1 \end{array} \right).$$

In the case $C_\pi = C_2^4$, denote by A_i^τ the $[44, 22, 8]$ code determined by $(I_2|A_i)$ and $C_\pi = \tau(G_1)$. There are 21 inequivalent codes, namely A_1^{id}, $A_2^{(2,5,7,3,6)}$, $A_3^{(2,5,6)}$, $A_3^{(2,5,7,3,6)}$, $A_3^{(2,5,4,6)}$, $A_4^{(2,5,7,3,6)}$, A_5^{id}, $A_6^{(2,5,6)}$, $A_6^{(4,5,6)}$, $A_6^{(4,5,7,8,6)}$, $A_6^{(3,5)(4,6)}$, A_7^{id}, $A_7^{(2,5,7,3,6)}$, $A_7^{(4,5,7,8)}$, $A_7^{(3,6,4,5,7)}$, A_8^{id}, $A_8^{(2,5,7,3,6)}$, A_9^{id}, $A_9^{(3,6,7)}$, $A_9^{(4,6,7,8)}$, and $A_9^{(2,5,6)}$. The code $A_2^{(2,5,7,3,6)}$ has an automorphism group of order 786839961600 and is equivalent to the code $C_{7,3}$ constructed above.

In the case $C_\pi = E_8$, denote by B_i^τ the $[44, 22, 8]$ code determined by $(I_2|A_i)$ and $C_\pi = \tau(G_2)$. There are 19 inequivalent codes, namely B_1^{id}, B_2^{id}, B_3^{id}, $B_3^{(5,6)}$, B_4^{id}, $B_4^{(3,7,6,8,5)}$, B_5^{id}, B_6^{id}, $B_6^{(6,7,8)}$, $B_6^{(5,6)}$, $B_6^{(4,5,6,7,8)}$, $B_6^{(3,7,8,6,4,5)}$, B_7^{id}, $B_7^{(6,7,8)}$, $B_7^{(5,6)}$, B_8^{id}, B_9^{id}, $B_9^{(5,6)}$, and $B_9^{(5,6,7)}$. The code B_2^{id} is equivalent to $C_{7,2}$, constructed in the previous section.

Case III: $\dim_{I_1} M_1 = \dim_{I_2} M_2 = 3$. Then

$$gen(M_1) = \left(\begin{array}{cccccc} e_1 & 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & e_1 & 0 & \alpha_4 & \beta_1 & \beta_2 \\ 0 & 0 & e_1 & \alpha_5 & \beta_3 & \beta_4 \end{array} \right),$$

where $\alpha_i \in \{0, e_1\}, i = 1, \cdots, 5$, and $\beta_i \in I_1, i = 1, 2, 3, 4$. There are 18 inequivalent codes M_1 with minimum weight ≥ 2. We can fix the generator matrices for M_1 and M_2 and consider all possibilities for C_π.

When $C_\pi = E_8$ we obtain 64 inequivalent codes with $W_{44,1}$ for $\beta = 10$, 17, 24, 31, 38, 52, and 122. In the case $C_\pi = C_2^4$ we obtain 87 inequivalent codes with $W_{44,2}$ for $\beta = 0$, 7, 14, 21, 28, 35, 42, 56, and 154. The codes with $\beta = 122$ and $\beta = 154$ are equivalent to $C_{7,1}$ and $C_{7,3}$, respectively.

Theorem 14 *There are exactly* 191 *inequivalent* $[44, 22, 8]$ *codes having an automorphism of order* 7*.*

Table 9: Automorphism groups of self-dual $[44, 22, 8]$ codes for $C_\pi = E_8$

$\vert Aut(C)\vert$	7	14	21	28	42	56	84	112	126	168
♯ codes	13	35	1	5	9	2	2	2	1	1
$\vert Aut(C)\vert$	252	336	672	1344	2688	5040	5376	8064	64512	3251404800
♯ codes	1	2	2	1	1	1	1	1	1	1

Table 10: Automorphism groups of self-dual $[44, 22, 8]$ codes for $C_\pi = C_2^4$

$\vert Aut(C)\vert$	7	14	28	42	56	112	168	336
Number of codes	49	33	4	1	3	2	1	2
$\vert Aut(C)\vert$	672	1344	2688	10752	21504	43008	786839961600	
Number of codes	1	2	3	1	1	3	1	

The orders of the automorphism groups of these codes are presented in Tables 9 and 10. The weight enumerators of the constructed codes are listed in Table 11.

5 Summary

The self-dual $[44, 22, 8]$ codes having automorphisms of order 11 are classified in [21] and [20]. The codes invariant under an automorphism of order 5 are presented in [3] and [4]. Summarizing these classifications and the results from the previous sections, we obtain the following theorem.

Theorem 15 *There are exactly* 395555 *inequivalent self-dual* $[44, 22, 8]$ *codes having an automorphism of odd prime order.*

All constructed codes with $\beta \geq 43$ have automorphisms of order 3. In Table 12 we list the number of codes having an automorphism of odd prime order according to their weight enumerators but only for these values of β for which there are also codes having automorphisms of order 5, 7 or 11, but not 3. For the other values of β the number of all extremal self-dual codes having an automorphism of odd prime order is the same as in Table

Table 11: Weight enumerators of self-dual $[44, 22, 8]$ codes having an automorphism of order 7

β in $W_{44,1}$			10	17	24	31	38	52	59	122
Number of codes			23	19	14	12	9	4	1	1
β in $W_{44,2}$	0	7	14	21	28	35	42	56	104	154
Number of codes	27	29	32	5	7	1	1	4	1	1

8. We can send the generator matrices of the obtained codes by e-mail to everybody who is interested.

Table 12: Self-dual $[44, 22, 8]$ codes having an automorphism of odd prime order

β	0	4	5	7	9	10	11	12
W_1	-	-	-	-	-	**1506**	1539	**397**
W_2	**54**	**23926**	**19293**	**42796**	**9658**	**43639**	**30237**	9070
β	14	15	17	19	20	21	22	24
W_1	2659	680	**2549**	3385	2182	851	**2561**	**820**
W_2	**20026**	**5672**	9965	**7909**	**5888**	**2445**	**4802**	**2117**
β	25	27	28	29	30	31	32	
W_1	1428	**528**	1051	558	431	**525**	**523**	
W_2	**2251**	978	**1470**	**872**	**852**	965	1048	
β	34	35	37	38	42	44		
W_1	396	201	**179**	**207**	**96**	115		
W_2	**1090**	**607**	477	526	221	**153**		

Looking at the tables, one can notice that there is only one code for $\beta = 154$. This code has a large automorphism group - its order is $2^{16} \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11^2 = 786839961600$. The same is the situation with the codes for $\beta = 122$ and $\beta = 104$. These two codes have automorphism groups of orders $2^{15} \cdot 3^4 \cdot 5^2 \cdot 7^2 = 3251404800$ and $2^{13} \cdot 3^4 \cdot 5^2 \cdot 7 = 116121600$, respectively. In Table 13 we present the orders of the automorphism groups of the codes with $\beta \geq 67$. All these orders are multiples of $288 = 9 \cdot 2^5$. Actually, all 12 codes with automorphism groups of orders bigger than 400000 have weight enumerators of both types with $\beta \geq 72$ and are given in Table 13. We list the number of codes C with full automorphism groups of orders $6000 < |Aut(C)| < 400000$, $|Aut(C)| \neq 2^s$, in Table 14. The code with the largest automorphism group (order 368640) which is not listed in Table 13 has weight enumerator $W_{44,1}$ with $\beta = 42$. Actually, the full automorphism group for most of the codes (exactly 309666) is the cyclic group of order 3. These codes have weight enumerators of both types with $\beta \leq 42$.

Looking at the weight enumerators of the extremal codes of length 44 constructed up to now, the following open problems arise:

1. Prove that there are not extremal self-dual $[44, 22, 8]$ codes with weight enumerator $W_{44,1}$ for $\beta = 69, 71, 73, 75, \ldots, 81, 83, 84, 85, 87, 88, 89, 91, \ldots, 121$, or $W_{44,2}$ for $\beta = 57, 63, 65, 67, 69, 71, 73, 75, 77, \ldots, 81, 83, 84, 85, 87, 88, 89, 91, \ldots, 103, 105, \ldots, 153$.

2. Are the constructed codes with weight enumerators $W_{44,1}$ for $\beta = 61, 63, 68, 72, 82, 86, 90, 122$, and $W_{44,2}$ for $\beta = 59, 61, 66, 68, 70, 86, 90, 104, 154$, the unique examples for their weight enumerators?

3. Which of these codes have connections with combinatorial designs?

Table 13: The orders of the automorphism groups of the self-dual $[44, 22, 8]$ codes with $\beta \geq 67$

β	67	68	70	72	74	76
♯ codes with $W_{44,1}$	2	1	2	1	2	-
$\|Aut(C)\|$	2592 2304	5184	13824 18432	6912	6912 73728	-
♯ codes with $W_{44,2}$	-	1	1	3	4	2
$\|Aut(C)\|$	-	207360	69120	92160 1105920 184320	69120 331776-2 14745600	207360 165888
β	82	86	90	104	122	154
♯ codes with $W_{44,1}$	1	1	1	-	1	-
$\|Aut(C)\|$	7372800	1105920	2211840	-	3251404800	
♯ codes with $W_{44,2}$	2	1	1	1	-	1
$\|Aut(C)\|$	663552 2211840	1105920	14745600	116121600	-	786839961600

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Table 14: Number of the self-dual $[44, 22, 8]$ codes with $6000 < |Aut(C)| < 400000$

$\|Aut(C)\|$	368640	331776	207360	184320	165888	98304	92160
Number of codes	1	2	2	1	1	1	1
$\|Aut(C)\|$	73728	69120	64512	61440	55296	46080	43008
Number of codes	2	2	1	2	1	1	3
$\|Aut(C)\|$	36864	34560	21504	18432	15552	13824	12288
Number of codes	4	2	1	8	1	2	11
$\|Aut(C)\|$	11520	10752	10368	9216	8064	6912	6144
Number of codes	1	1	1	6	1	6	35

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